SEMILATTICES OF ARCHIMEDEAN SEMIGROUPS

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“Semilattices of Archimedean Semigroups”

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2006 - Scientific Monograph of National Importance

(Ministry of Education, Science and Technological Development of Republic of Serbia)

Presents some **results** of investigations concerning certain types of **subsets** and **subsemigroups** of a given semigroup in order to describe its structure.

Semigroup $(S, \cdot)$

set S together with an associative binary operation $\cdot$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad \text{for any} \quad x, y, z \in S$$

- Where nature of the multiplications is clear from the context, it is written S rather than $(S, \cdot)$

- Frequently, xy is written rather than $x \cdot y$

- In the history of mathematics, the algebraic theory of semigroups is a relative new-comer, with the theory proper developing only i**n the second half of the twentieth century.**

- Historically, it can be viewed as an algebraic apstraction of the properties of the composition of transformations on a set.

- Late sources include an abstraction of certain ideas arising in connection with topological or linear spaces, differential geometry,

- ... But, there is no daught about it, the main sources came from **group theory and ring theory** - much groundwork was laid by researchers arriving at the study of semigroups from the directions of both group and ring theory.

Of all generalizations of the group and ring concepts the semigroup is the one that has attracted the most interest by far.

C. Hollings: *The early development of the algebraic theory of semigroups*, Archive for History of Exact Science, 63(5) (2009), 497-536.

1160 distinct of order 5
15 793 distinct of order 6
836 021 distinct of order 7
1 843 120 128 distinct of order 8

⋮

(neither isomorphic nor anti-isomorphic)

• Green's relations, the fundamental tools in the structure theory of semigroups, **do not have nontrivial analogues** in groups, rings, quasigroups, lattices, universal algebras, fields.

Semigroups

- DO NOT much reseamble groups and rings
- DO NOT much reseamble any other algebraic structures

- Ring $(R, +, \cdot) \longrightarrow (R, \cdot)$ is a semigroup

OPEN QUESTION: conditions for a semigroup to be the multiplicative semigroup of a ring

- Congruences

– groups: uniquely determined by its normal subgroups
– rings: there is a bijeetion between the congruenees and the ideals of a ring

– **semigroups**: no such device is available - one must study congruences as such.

Semigroup theory has developed special methods and new semigroup classes have come into the center of interest.

TO CONCLUDE

- their unique character
- their diversity
- often complex structure

MAKE SEMIGROUPS INTERESTING

Describing semigroup and its structure is a formidable task.

Various approaches have been developed over the years to construct

frameworks

for understanding the structure of semigroups.

Introductional and basic study of semigroups (of any kind) includes a description of

- special subsets
- special elements
- ideals
- Green's relations
- homomorphisms
- congruences

Special elements and subsets

- **Regular subsets** of S:

  $\boldsymbol{E(S)} \;=\; \{\, a \in S \,|\, a = a^2 \,\}$
  (set of all idempotents of S)

  $\boldsymbol{Reg(S)} \;=\; \{\, a \in S \,|\, (\exists x \in S)\; axa = a \,\}$ - the regular part of S
  (set of all regular elements of S)

  $\boldsymbol{Gr(S)} \;=\; \{\, a \in S \,|\, (\exists x \in S)\; axa = a \,\&\, ax = xa \,\}$ - the group part of S
  (set of all completely regular or group element of S)

i

S is *band*: $\boldsymbol{S = E(S)}$

S is *regular*: $\boldsymbol{S = Reg(S)}$

S is *completely regular*: $\boldsymbol{S = Gr(S)}$

S is $\boldsymbol{\pi}$-*regular*: $(\forall a \in S)(\exists k \in \mathbb{N})\ \boldsymbol{a^k \in Reg(S)}$

S is *completely* $\boldsymbol{\pi}$-*regular*: $(\forall a \in S)(\exists k \in \mathbb{N})\ \boldsymbol{a^k \in Gr(S)}$

S is *periodic*: $(\forall a \in S)(\exists k \in \mathbb{N})\ \boldsymbol{a^k \in E(S)}$

ii

In general

$$\boldsymbol{E}(S) \subseteq \boldsymbol{Reg}(S) \subseteq \boldsymbol{Gr}(S)$$

Theorem 0.1 ***Shevrin-Veronesi***
A semigroup S is uniformly π-regular (a semilattice of completely archimedean semigroups) if and only if S is π-regular and $\boldsymbol{Reg}(S) = \boldsymbol{Gr}(S)$.

- **Regular parts** of **subsemigroup $\boldsymbol{T}$** of S:

  $\boldsymbol{Reg(T)} \;=\; \{\, a \in T \mid (\exists x \in T)\; axa = a \,\}$
  regular part of T;

  $\boldsymbol{reg(T)} \;=\; \{\, a \in T \mid (\exists x \in S)\; axa = a \,\}$
  s-regular part of T.

In general

$$\boldsymbol{Reg(T) \subseteq reg(T) = T \cap Reg(S)}.$$

Problem: Semigroup S with

$$\boldsymbol{Reg(T) = reg(T)}$$

for every subsemigroup T of S.

* * *

– Equality (known up to the date):

T is two-sided ideal of S

Theorem 0.2 ***Shevrin***

Let S be a completely π-regular and let T be its subsemigroup which is itself completely π-regular, then

$$Gr(T) = gr(T).$$

- **Ideals** (left, right, two-sided, quasi-, bi-, principal).

— Let $\boldsymbol{A}$ be a nonempty subset of S

A is a *left* (*right*) *ideal* of S: $\boldsymbol{SA \subseteq A}$ $(AS \subseteq A)$

A is a (*two-sided*) *ideal* of S: $\boldsymbol{SA \cup AS \subseteq A}$

A is a *bi-ideal* of S: $\boldsymbol{ASA \subseteq A}$

— Let $\boldsymbol{a}$ be an element of a semigroup S

$\boldsymbol{J(a)} = a \cup aS \cup Sa \cup SaS$ - *principal ideal* of S *generated by element* a

$\boldsymbol{L(a)} = a \cup Sa$ $\quad$ $L(e) = \boldsymbol{Se}$

$\boldsymbol{B(a)} = a \cup aSa$ $\quad$ $B(e) = \boldsymbol{eSe}$

$\boldsymbol{eSe}$ - *local submonoid* or *local monoid* of S

* * *

Possible solution: relationship between regular parts of ideals

Solution can be reached starting from

divisability

– In oridinary arithmetics we often interested whether one number divides another

– Divisability in semigroups can be considered similarly

– In general, multiplication in semigroup is not commutative

– There are three different kinds of divisability

$$a \mid b \;\Leftrightarrow\; b \in J(a), \quad a \underset{l}{\mid} b \quad\Leftrightarrow\quad b \in L(a), \quad a \underset{r}{\mid} b \;\Leftrightarrow\; b \in R(a),$$

$$a \underset{t}{\mid} b \quad\Leftrightarrow\quad a \underset{l}{\mid} b \;\&\; a \underset{r}{\mid} b$$

Divisability relations are among the most important quasi-orderings defined on semigroup S

- *Green's equivalences:*

$$a\ \boldsymbol{\mathcal{J}}\ b \quad \Leftrightarrow \ J(a) = J(b) \ \Leftrightarrow \ a \mid b \ \&\ b \mid a$$

$$a\ \boldsymbol{\mathcal{L}}\ b \quad \Leftrightarrow \ L(a) = L(b) \ \Leftrightarrow \ a \underset{l}{\mid} b \ \&\ b \underset{l}{\mid} a$$

$$a\ \boldsymbol{\mathcal{R}}\ b \quad \Leftrightarrow \ R(a) = R(b) \ \Leftrightarrow \ a \underset{r}{\mid} b \ \&\ b \underset{r}{\mid} a$$

$$\boldsymbol{\mathcal{H}} \ = \ \mathcal{L} \cap \mathcal{R}$$

$$\boldsymbol{\mathcal{D}} \ = \ \mathcal{L}\mathcal{R} \ = \ \mathcal{R}\mathcal{L}$$

- **Green's subsets** of S:

$\boldsymbol{\mathcal{U}_{\mathcal{T}}(S)}$ - union of all $\mathcal{T}$-classes of S which are subsemigroups

$\boldsymbol{\mathcal{L}^{S}_{\mathcal{T}}(S)}$ - union of all $\mathcal{T}$-classes of S which are left simple semigroups

$\boldsymbol{\mathcal{L}_{\mathcal{T}}(S)}$ - union of all $\mathcal{T}$-classes of S which are left groups

$\boldsymbol{\mathcal{H}_{\mathcal{T}}(S)}$ - union of all $\mathcal{T}$-classes of S which are groups

$\mathcal{T}$ is one of Green's relations

We consider **relations** between:

– different (generalized) regular subsets of a semigroup S;

– (generalized) regular and Green's subsets of S;

– (generalized) regular parts of principal ideals of S generated by idempotents;

– (generalized) regular parts of subsemigroups of S.

As results we obtained **connections** with:

– *semilattice decompositions* of semigroups;

– *local properties* of semigroups;

– *hereditary properties* of semigroups.

- **Chapter 1**

consists of a preliminary material needed for reading the main parts of the book.

- **Chapter 2**

properties of semigroups with the non-empty set of idempotents based on connections between the **group** and **two regular parts of principal ideals** of these semigroups, generated by idempotents.

- **Chapter 3**

connections of

semilattice decompositions

of semigroups with certain equalities between **generalized regular** and **Green's subsets**

- **Chapter 4**

results concerning connections of equalities between different (generalized) regular subsets of semigroups with their

local properties.

- **Chapter 5**

equalities of (generalized) regular subsets of semigroups and their connections with

hereditary properties

of semigroups.

Chapter 2

Regular subsets of a semigroup

Semigroups with

$$Reg(T) = reg(T)$$

T runs over one of the following families of subsemigroups

- $\{Se \mid e \in E(S)\}$
- $\{eS \mid e \in E(S)\}$
- $\{eSf \mid e, f \in E(S)\}$

Theorem 0.3 *The following conditions on a semigroup* S*,* $E(S) \neq \varnothing$*, are equivalent:*

(i) $(\forall e \in E(S))\ reg(Se) = Gr(Se)$

(ii) $(\forall e \in E(S))\ reg(Se) = Reg(Se)$

(iii) $(\forall e \in E(S))\ reg(Se) \subseteq LReg(Se)$

(iv) $Reg(S) \subseteq LReg(S)$

(v) $Reg(S) = Gr(S)$

Corollary 0.1 *Each of the following conditions on a semigroup* S *is equivalent to the above conditions (i)–(v):*

(vi) $(\forall e, f \in E(S))\ reg(eSf) = Gr(eSf)$;

(vii) $(\forall e, f \in E(S))\ reg(eSf) = Reg(eSf)$;

(viii) $(\forall e, f \in E(S))\ reg(eSf) \subseteq LReg(eSf)$.

Chapter 3

Decompositon of (quasi-)π-semisimple semigroups

- Connections between:

– **semilattice decomposition** of of semigroup **into archimedean components**

– **equalities** between

(generalized) regular subsets and
Green's subsets

• **Method of decompositions of semigroups** - based on partition of the semigroup, describing the structure of each components and establishing the connections between them.

. **Band decompositions** - S/ϱ is a band and each ϱ-class is a subsemigroup of S.

– A. H. Clifford, 1941, 1954.

. **Semilattice decompositions** of semigroups

– introduced by A. H. Clifford (1941),

– special contributions: T. Tamura, N. Kimura and J. Shafer
— the series of their papers began in 1954, where semilattice decompositions of commutative semigroups were considered
— of great importance: T. Tamura (1956, 1964)

– – ***Any semigroup is a semilattice of semilattice-indecomposable semigroups.***

– M. Petrich , M. S. Putcha, R. Šulka, M. Ćirić, S. Bogdanović, M. Mitrović

- **Archimedean semigroups**: semigroups in which for any two elements one of them divides some power of the other

$$(\forall a, b \in S)(\exists k \in \mathbb{N}) \quad \boldsymbol{a^k \in SbS}$$

– introduced indenpendetly by T. Tamura and N. Kimura in 1954 (from their studying the semilattice decompositions of commutative semigroups) and G. Thierrin (1954)

Proposition 0.1 ***T. Tamura*** *(1972)*
An archimedean semigroup is a semilattice-indecomposable.

. Semilattice of archimedean semigroups or **Putcha's semigroups**

– T. Tamura (1954): any commutative semigroup is a semilattice of archimedean semigroups (commutativity means that components are t-archimedean)

– J. L. Chrislook (1969): result was extended to the class of medial semigroups

– T. Tamura, J. Shafer (1972): the class of exponential semigroups

♦ M. S. Putcha (1973): first complete description

3.2. Semilattices of Nil-exstensions of Simple Regular Semigroups

S is $\boldsymbol{\pi}$-*regular*: $(\forall a \in S)(\exists k \in \mathbb{N})$ $\boldsymbol{a^k \in Reg(S)}$

- Concept of **π-regularity** appeared first in ring theory - McCoy [1939]

∼ In semigroup theory, attracted great attention both as
- a generalization of regularity
- a generalization of finiteness and periodicity

∼ Was studied under different names:
- quasi-regularity (M. Putcha, J. L. Galbiati, M. L. Veronesi)
- power-regularity and π-regularity (S. Milić, S. Bogdanović, M. Ćirić, M. Mitrović)
- eventual regularity (D. Easdown, R. Edwards, P. Higgins)

Lemma 0.1 *The following conditions on a semigroup S are equivalent:*

(i) *S is regular and simple*

(ii) *S is simple and π-regular*

(iii) $(\forall a, b \in S) \qquad a \in aSbSa$

Theorem 0.4 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of nil-extensions of simple regular semigroups;*

(ii) $\forall a, b \in S)(\exists n \in \mathbb{N})\ (ab)^n \in (ab)^n Sa^2 S(ab)^n$*;*

(iii) *S is π-regular and*

(iv) *S is π-regular and $\boldsymbol{Reg(S)} = \boldsymbol{Intra(S)} = \boldsymbol{\mathcal{U}_{\mathcal{J}}(S)}$.*

* * *

M. Mitrović, S. Bogdanović, M. Ćirić (1997)

3.5. Semilattices of Completely Archimedean Semigroups (uniformly π-regular semigroups)

S is <u>*completely*</u> $\boldsymbol{\pi}$-<u>*regular*</u>: $(\forall a \in S)(\exists k \in \mathbb{N})\ \boldsymbol{a^k \in Gr(S)}$

· **pseudoinverse** of an element, essential for completely π-regular semigroups - Drazin [1958]

·· the first investigations of such semigroups - W. D. Munn [1968]

·· different names trough literature:
- *pseudo-invertible* - W. D. Munn
- quasi-completely regular - M. S. Putcha
- completely quasi-regular - J. L. Galbiati, M. L. Veronesi
- groupbound - B. L. Madison, T. K. Mukherjee, M. K. Sen, P. M. Higgins
- epigroups - L. N. Shevrin

– ***completely archimedean***: archimedean and completely π-regular - the most popular subclass of the class of archimedean semigroups

Theorem 0.5 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of completely archimedean semigroups (uniformly π-regular)*

(ii) *S is π-regular and $Reg(S) = Gr(S)$*

(iii) *S is completely π-regular and* $\boldsymbol{Gr(S) = \mathcal{U}_{\mathcal{J}}(S) = \mathcal{U}_{\mathcal{L}}(S)}$

Chapter 5

Hereditary Properties of Semilattices of Archimedean Semigroups

- S is a *hereditary* $\mathcal{K}$-*semigroup* if each its subsemigroup belongs to $\mathcal{K}$ ($\mathcal{K}$ - a class or property of semigroups)

- **main intention:** to give an interesting bridge between:
  - equalities of regular parts of subsemigroups
  - hereditary properties of semigroup

- T. Tamura (1975): the class of archimedean and the class of semilattices of archimedean semigroups **are not** subsemigroup closed

S. Bogdanović, M. Cirić, M. Mitrović (1995)

<u>*Hereditary archimdean semigroup*</u>: every its subsemigroup is archimedean

Theorem 0.6 *The following conditions on a semigroup S are equivalent:*

(i) *S is archimedean and periodic*

(ii) *S is hereditary archimedean and contains a primitive idempotent*

(iii) $(\forall a, b \in S)(\exists k \in \mathbb{N})\ a^k = (a^k b^k a^k)^k$

. **The greatest subsemigroup closed** subclass of the class of semilattices of archimedean semigroups

Theorem 0.7 *A semigroup S is a hereditary semilattice of archimedean semigroups if and only if for all $a, b \in S$ there exists $k \in \mathbb{N}$ such that*

$$(ab)^k \in \langle a, b \rangle\, a^2 \,\langle a, b \rangle.$$

Completely hereditary archimedean: hereditary archimedean and completely π-regular

Theorem 0.8 *The following conditions on a semigroup S are equivalent:*

(i) *S is a semilattice of completely hereditary archimedean semigroups*

(ii) $(\forall a, b \in S)(\exists k \in \mathbb{N})\ (ab)^k = (ab)^k((ba)^k(ab)^k)^k$

(iii) *S is periodic and $Reg(S) = Gr(S)$*

(iv) *S is* ***hereditary uniformly π-regular***

Theorem 0.9 *A semigroup S is hereditary uniformly π-regular if and only if*

$$\boldsymbol{reg(T) = Reg(T) \neq \varnothing},$$

for each subsemigroup T of S.

THANK YOU FOR YOUR ATTENTION!